ON TWISTED CUBIC CURVES THAT HAVE A DIRECTRIX*

BY

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Introduction.

The species of twisted cubic parabola having a directrix has been examined by BÖKLEN,† and FRANZ MEYER.‡ The cubic parabola is a twisted cubic having among its osculating planes the plane at infinity. It is said to have a directrix when the osculating planes can be arranged in triads, each triad consisting of three faces of a trirectangular triedral angle. It is shown that the locus of vertices of such angles is a straight line, and that line is called the directrix of the curve. The further question is now to be considered, whether also any cubics that are not parabolas may possess a directrix; that is, whether the osculating planes of a cubic ellipse or hyperbola may ever be grouped in sets of three mutually perpendicular, and if so, what is then the locus of the vertex where three perpendicular planes intersect.

It will be shown that two conditions are necessary to the existence of a directrix when the cubic is not parabolic, and that such a directrix must be a straight line. Thus will be established by metrical characteristics a second class, more extensive than the species of cubics with directrix already known, inasmuch as the latter have to satisfy three conditions. Noteworthy is the fact that this second species does not include all parabolas of the first, and that together they comprise all possible cubics having a directrix.

As an auxiliary it is necessary to consider the system of polar triangles of a conic — the conic absolute of metrical geometry — and to discuss a new simultaneous covariant of the ternary cubic and quadric (§ 3), whose vanishing indicates that an infinite number of polar triangles of the conic are inscribed in the cubic. Incidentally there is noticed a transformation of this covariant quadric into itself by a group of non-linear substitutions of the parameters upon which it depends; it is in fact a semi-combinant of itself and the cubic.

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† Schlömilch's Zeitschrift, vol. 29 (1884), pp. 378–382.

‡ Böklen's Mittheilungen, vol. 1 (1884), pp. 11–16; or Schlömilch's Zeitschrift, vol. 30 (1885), pp. 345–349.

§ 1. *The developable of a twisted cubic.*

The osculating planes of a twisted cubic form a developable of order 4 and class 3; that is, through every point of space there pass three planes of the system. Hence the plane at infinity, if not itself one of the system, cuts the developable in a curve of order 4 and class 3, a quartic curve with three cusps. Now two planes are perpendicular when their traces at infinity are conjugate lines with respect to the circle common to all spheres. If then three planes are to be mutually perpendicular, their traces at infinity are required to form a self-conjugate or polar triangle with respect to that circle, the absolute of euclidean metric. The problem of finding points in which three planes of the system meet at right angles becomes therefore the problem of finding polar triangles of a conic, whose sides shall be tangents of a tricuspidal quartic curve in the same plane. If there can be an infinite number of such triads of planes, then also the corresponding circumscribed polar triangles must be infinite in number. But in place of this problem we may substitute its dual equivalent, to find how many polar triangles of a given conic are inscribed in a nodal cubic in the same plane; and under what conditions the number of such triangles may become infinite.

In the special case where the plane at infinity itself is an osculating plane of the cubic, its section with the developable is reduced to a doubly counting tangent line and a conic. The problem for this case reduces to the familiar one, to find how many polar triangles of the absolute circle are circumscribed to (or inscribed in) an arbitrary conic, and under what conditions this member is infinite. This is the case of the cubical parabola, already treated from this point of view by W. Franz Meyer (loc. cit.). I shall review first this simpler case, as exhibiting the method to be employed in the other.

§ 2. *Polar triangles of one conic inscribed in another. Application to the cubical parabola.*

Denote by a and α the absolute conic and any second, and let their equations be

$$a_x^2 = b_x^2 = 0\,, \qquad \alpha_x^2 = 0\,.$$

In α there is to be inscriptible a triangle self-conjugate with respect to a. Each side of such a triangle is a line divided harmonically by the two conics, and is therefore a tangent: $u_x = 0$, to the envelope represented by the equation *

$$(a\alpha u)^2 = 0\,.$$

*Clebsch Lindemann, *Vorlesungen über Geometrie*, vol. I, p. 276.

Let the opposite vertex of the triangle be a point (y). The equation of its polar, the side in question, is

$$a_y a_x \equiv b_y b_x \equiv c_y c_x \equiv u_x = 0.$$

Therefore the u's have the values:

$$u_1 = a_y a_1, \qquad u_2 = a_y a_2, \qquad u_3 = a_y a_3.$$

Substituting these in the equation of the envelope, we have for the locus of the pole (y) the equation:

$$(a\alpha b)(a\alpha c) b_y c_y = 0.$$

This reduces identically to the form

$$\tfrac{1}{2}(ab\alpha)^2 c_y^2 - \tfrac{2}{3}(abc)^2 \alpha_y^2 = 0.$$

From this it appears that the conic α will itself be the locus of poles of lines harmonically divided, if

$$(ab\alpha)^2 = 0.$$

By duality we have therefore the theorem:

If the trace of the developable of a cubical parabola on the plane at infinity, considered as a class conic, is apolar to the circular absolute considered as an order conic, then all the osculating planes of the cubical parabola can be grouped into triads of mutually perpendicular planes.

Call the locus of the intersection of such a triad the directrix of the cubic. From the nature of the problem this directrix must be an algebraic curve. Its order can be found by noting in how many points it meets an arbitrary plane, for example one of the osculating planes of the system. Through each intersection of this plane with the directrix there pass only three osculating planes, one of which is the plane in question. Hence if the directrix be of order n, this plane must be a component in n different triads; so that its trace at infinity is a side in each of n different polar triangles of the conic absolute, all inscribed in a second conic. But one line has only one pole with respect to the absolute conic, and meets the second conic in only two points, therefore it can form part of only one such polar triangle. *Accordingly* $n = 1$, *and the directrix must be a straight line.*

§ 3. *Polar triangles of a conic, inscribed in a plane cubic.*

The foregoing section may serve as model in the examination of a cubic in the plane of the conic absolute. First there is to be found the equation of the envelope of a line divided harmonically by its intersections with the conic and two of its three intersections with the cubic. The locus of the pole of this line is derivable from the envelope, as it was in the case of a second conic. Finally the conditions must be analyzed under which the locus of the pole contains all points of the cubic.

Denote a binary quadric and cubic respectively by $a_x^2 \equiv b_x^2 \equiv c_x^2$, and $\alpha_x^3 \equiv p_x^2 \pi_x \equiv q_x^2 \rho_x$. Two roots of the cubic will be harmonic to the roots of the quadric if either the quadrics a_x^2 and p_x^2 are apolar, or the polar of π_x with respect to the quadric a_x^2 is one factor of p_x^2; i. e., if the compound condition is satisfied:

(1) $$G \equiv (ap)^2 (b\pi)(bq)(c\rho)(cq) = 0.$$

To express this condition in terms of invariants of a_x^2 and α_x^3, reduce the latter* to invariants of a_x^2, p_x^2 and π_x, when comparison shows that

(2) $$G = F + \tfrac{1}{4} D \cdot E \equiv \tfrac{1}{4}(a\alpha)(a\beta)\{(bc)^2(\alpha\beta)^2 + 4(b\alpha)^2(c\beta)^2\}.$$

Hence by bordering we obtain the condition for a line, $u_x = 0$ (ternary) to meet in such sets of points a conic and cubic curve given by the ternary equations

$$a_x^2 = 0, \qquad \alpha_x^3 = 0.$$

The condition is

(3) $$G(u) \equiv (a\alpha u)(a\beta u)\{(bcu)^2(\alpha\beta u)^2 + 4(b\alpha u)^2(c\beta u)^2\} = 0.$$

Identify the line (u) with the polar of a point y,

$$u_x \equiv a_y a_x \equiv d_y d_x \equiv e_y e_x \equiv \cdots \equiv k_y k_x,$$

so that the envelope (3) gives as its reciprocal with respect to the conic a sextic curve:

$$G(u) \equiv \Gamma(y) \equiv (a\alpha d)(a\beta e)\left\{\begin{matrix}(bcf)(bcg)(\alpha\beta h)(\alpha\beta k) \\ +4(b\alpha f)(b\alpha g)(c\beta h)(c\beta k)\end{matrix}\right\} d_y e_y f_y g_y h_y k_y = 0.$$

Recalling the harmonic section upon the line (u) and the definition of the point (y) as its pole, we have already this result:

THEOREM.—*The 18 points in which the sextic:* $\Gamma(y) = 0$, *meets the cubic:* $\alpha_x^3 = 0$, *are vertices of triangles, proper or improper, self-conjugate with respect to the conic:* $a_x^2 = 0$.

Geometrically it is evident that the condition imposed on (y) is doubly satisfied by each intersection of the cubic with the conic, for the polar of such a point is there tangent to the conic, and both the remaining intersections of this tangent with the cubic count as conjugate to its point of contact. These six intersections are therefore double points on the sextic. This indicates a reduction of the form $\Gamma(y)$ to a sum of terms of the second degree (or higher) in the

* See CLEBSCH, Binäre Formen, p. 209.

forms a_x^2, α_x^3. Effecting this reduction by the usual mode, we can express the sextic covariant as follows, D denoting the discriminant $(abc)^2$ of the quadric:

$$
\begin{aligned}
(4)\qquad \Gamma(y) \equiv & -\tfrac{1}{54} D^3 \cdot \alpha_y^3 \beta_y^3 + \tfrac{1}{6} D^2 . (ab\alpha)^2 \alpha_y \cdot c_y^2 \beta_y^3 \\
& + \tfrac{1}{18} D^2 \cdot (a\alpha\beta)^2 \alpha_y \beta_y \cdot b_y^2 c_y^2 \\
& - \tfrac{1}{2} D \cdot (ab\alpha)(ab\beta)(cd\alpha)^2 \beta_y^2 \cdot e_y^2 f_y^2 \\
& - \tfrac{1}{3} D \cdot (ab\alpha)^2 (cd\beta)^2 \alpha_y \beta_y \cdot e_y^2 f_y^2 \\
& + \tfrac{1}{2} (ab\alpha)(ab\beta)(cd\alpha)^2 (ef\beta)^2 \cdot g_y^2 h_y^2 k_y^2 \\
& + \tfrac{1}{6} D \cdot (ab\alpha)(ab\beta)(c\alpha\beta)^2 \cdot d_y^2 e_y^2 f_y^2 .
\end{aligned}
$$

The terms of this reduced covariant free from a factor α_y^3 do not all contain a_y^2 to a degree higher than the second, therefore the 18 intersections of cubic and sextic lie, 12 in 6 double-points on the fundamental conic, and 6 in other points upon a covariant conic whose equation is

$$
\begin{aligned}
(5)\qquad \Gamma'(y) \equiv & D^2 \cdot (a\alpha\beta)^2 \alpha_y \beta_y - 9 D \cdot (ab\alpha)(ab\beta)(cd\alpha)^2 \beta_y^2 \\
& - 6 D \cdot (ab\alpha)^2 (cd\beta)^2 \alpha_y \beta_y + 9 (ab\alpha)(ab\beta)(cd\alpha)^2 (ef\beta)^2 \cdot g_y^2 \\
& + 3 D \cdot (ab\alpha)(ab\beta)(c\alpha\beta)^2 \cdot d_y^2 = 0 .
\end{aligned}
$$

THEOREM.—*The conic $\Gamma'(y) = 0$ meets the cubic in two sets of three points, each set being the vertices of a triangle self-conjugate with respect to the conic absolute, $a_x^2 = 0$.*

That two such determinate triangles exist in the general case may be verified upon a degenerate cubic consisting of three lines, by the use of projective ranges on the three lines. Or the cubic may be taken to be any line together with any conic not apolar to the conic absolute and not containing the pole of the line.* But we have reached the peculiar aim of this inquiry in demonstrating the following

THEOREM.—*In order that every point on the cubic may be a vertex of one inscribed triangle, self-conjugate with respect to the conic absolute, the quadric covariant $\Gamma'(y)$ must vanish identically; this is also the sufficient condition.*

For when $\Gamma'(y)$ vanishes identically, the sextic (4) degenerates into two cubics:

$$\Gamma(y) = \frac{D^2}{54} \{ 9 (ab\alpha)^2 \alpha_y c_y^2 - (abc)^2 \cdot \alpha_y^3 \} \cdot \beta_y^3 = \frac{D^2}{54} \cdot \phi \cdot \alpha .$$

This indicates the existence of two systems of polar triangles, the first having one vertex variable on the covariant cubic $\phi = 0$,† while two vary upon the

* One other relation to be avoided will be noticed at the end of this §.

† It may be remarked that this relation of ϕ and α is not reciprocal; indeed they are the first and second of an interminable succession, whose limit curve is given by the equation $\phi + \alpha = 0$.

cubic α; the second system having, as was required, all three vertices simultaneously variable upon the cubic α.

The identical vanishing of $\Gamma'(y)$ would appear to involve six conditions; but these cannot be all independent, since the quadric $\Gamma'(y)$ can be reduced to a sum of not more than three terms, giving at most three conditions. To calculate these explicitly, take as reference triangle one of the two polar triangles that are certainly inscribed in the cubic. The two equations then become:

$$a_x^2 = x_1^2 + x_2^2 + x_3^2,$$

$$\alpha_x^3 = 3\left[\alpha_1 x_3^2 x_1 + \alpha_2 x_1^2 x_2 + \alpha_3 x_2^2 x_3 + \beta_1 x_1 x_2^2 + \beta_2 x_2 x_3^2 + \beta_3 x_3 x_1^2\right] + 6\gamma x_1 x_2 x_3.$$

Inasmuch as the conic Γ' must contain the three vertices of this special triangle of reference, three terms disappear and our condition reduces to this:

$$(6) \qquad (\alpha_2\alpha_3 + \beta_2\alpha_3 + \beta_2\beta_3)y_2y_3 + (\alpha_3\alpha_1 + \beta_3\alpha_1 + \beta_3\beta_1)y_3y_1 + (\alpha_1\alpha_2 + \beta_1\alpha_2 + \beta_1\beta_2)y_1y_2 \equiv 0.$$

These three coefficients equated to zero give but two conditions, for

$$(7) \qquad \alpha_1(\alpha_2\alpha_3 + \beta_2\alpha_3 + \beta_2\beta_3) + \beta_3(\alpha_1\alpha_2 + \beta_1\alpha_2 + \beta_1\beta_2) \equiv (\alpha_2 + \beta_2)(\alpha_3\alpha_1 + \beta_3\alpha_1 + \beta_3\beta_1);$$

otherwise it may be seen that any one of the three is the eliminant of the other two. Any net of cubics circumscribed to this same reference triangle will contain three that satisfy these three conditions. This, therefore, is a double condition of third order.

When the conditions (6) are satisfied, can every point of the cubic be at once a vertex of two distinct polar triangles? Evidently not unless the polar of every point be a tangent to the curve. This can be proved impossible; but even if it could be admitted, the two polar triangles would coincide and constitute in fact but one. The salient facts may be collected in the following theorem:

THEOREM: *In any doubly infinite net of plane cubics circumscribed to a polar triangle of any proper conic there are three cubics which contain each an infinity of inscribed polar triangles of that conic. In any such cubic, each point is a vertex of one and only one inscribed polar triangle.* This is not to be understood as asserting that a net can contain only three such special cubics. In fact, since conditions (6) do not involve the coefficient γ, by allowing γ to vary alone a sheaf of cubics may be generated, all satisfying these conditions.

Of course this theorem applying to cubics both singular and non-singular is convertible by duality into a theorem for sextics with nine cusps, or tricuspidal quartics, and circumscribed polar triangles. That dual theorem is the one here to be applied.

If the cubic is degenerate, consisting of a conic K and a line L, it can satisfy conditions (6) only in two cases. First, the conic may be apolar to the quadric absolute and contain the pole P of the line L; then one set of inscribed polar triangles all have a vertex at P, while both remaining vertices lie upon L. Second, the conic K and the absolute conic a may lie in involution with respect to a point P, the pole of L; then the inscribed polar triangles will have one vertex variable upon L, while the other vertices form pairs in involution upon the conic K, with P for center of involution. The third conceivable alternative, that K alone should satisfy the condition of apolarity (§ 2), is not sufficient, as one verifies readily by applying condition (6) to the degenerate form:

$$\alpha_x^3 \equiv 3(p_1 x_2 x_3 + p_2 x_3 x_1 + p_3 x_1 x_2)(u_1 x_1 + u_2 x_2 + u_3 x_3).$$

This gives one negative theorem immediately, and a positive one by applying the theory of continuity.

THEOREM.—*The special conics of* § 2 *combined with arbitrary lines cannot always be considered degenerate cubics of the class defined by conditions* (6).

THEOREM.—*If the real cubic satisfying conditions* (6) *has an odd branch and an even branch, the even branch may contain one, two, or three vertices of the variable inscribed polar triangle.*

§4. *The covariant* $\Gamma'(y)$ *as a semicombinant.*

A special feature of this covariant $\Gamma'(y)$ is quite worthy of note. Remember that it intersects the cubic in the vertices of two polar triangles of the absolute. Now a triple infinity of cubics can be circumscribed to those same two triangles. Their equations are:

$$\alpha_x'^3 \equiv \alpha_x^3 + (u_1 x_1 + u_2 x_2 + u_3 x_3)\Gamma'(x) = 0. \tag{8}$$

Evidently now all these must have the same covariant conic: $\Gamma'(y) = 0$, for no other conic contains those six points. *Hence* $\Gamma'(x)$ *is a semi-combinant of the cubic* α_x^3 *and the conic* $\Gamma'(x)$. Otherwise stated, the conic $\Gamma'(x)$ is transformed into itself by a group of ∞^3 substitutions on the coefficients of α_x^3, viz.:

$$\alpha_1' = \alpha_1 + u_3(\alpha_3\alpha_1 + \beta_3\alpha_1 + \beta_3\beta_1), \text{ etc.},$$

$$\beta_1' = \beta_1 + u_2(\alpha_1\alpha_2 + \beta_1\alpha_2 + \beta_1\beta_2), \text{ etc.}$$

It is easily verified that by this substitution $\Gamma'(y)$ is reproduced with a multiplicative modulus which is quadric in the parameters u_1, u_2, u_3. Writing $\Gamma'(y)$ as dependent on α:

$$(9) \qquad \Gamma'(\alpha, y) = p_1 y_2 y_3 + p_2 y_3 y_1 + p_3 y_1 y_2,$$

we find after the substitution (8):

$$(10) \qquad \Gamma'(\alpha', y) \equiv \Gamma'(\alpha, y) \cdot \{ 1 + \Sigma u_1(\alpha_1 + \beta_1) + \Sigma u_2 u_3 p_1 \}.$$

The three quantities p_1, p_2, p_3 are thus automorphic invariants of this group; and a fourth is seen from the identities (7), viz:

$$(11) \qquad \alpha_1 p_1 - \beta_2 p_2 \equiv \alpha_2 p_2 - \beta_3 p_3 \equiv \alpha_3 p_3 - \beta_1 p_1 \equiv \alpha_1 \alpha_2 \alpha_3 - \beta_1 \beta_2 \beta_3.$$

This group and its geometrical representations invite further investigation though they are of no direct value for our present purpose.

§ 5. *Elliptic or hyperbolic twisted cubics, with a directrix.*

The trace at infinity of all osculating planes of a twisted cubical ellipse or hyperbola is a curve of class 3 and order 4, a tricuspidal quartic. If its tangents form an infinite system of polar triangles of the circular absolute, the osculating planes must form triads mutually perpendicular; and each plane can belong in only one triad, according to the theorem of § 3. Therefore the locus of the intersecting point of the planes of a triad can meet any one osculating plane in only one point, *hence this locus is a right line.* Such a line, when it exists for any particular cubic, may be termed the *directrix* of the curve. Its resemblance, however, is to the directrix of a plane parabola, not that of a plane ellipse or hyperbola, since these latter are circles. This fact may render the class of twisted cubics now under discussion even more interesting than the cubical parabolas with directrix,* as being a less obvious analogue to any plane curve. To sum up: *A twisted cubic whose developable satisfies at infinity the two independent (three simply related) conditions dual to* (6) *of* § 3 *relative to the circular absolute has its osculating planes grouped into triads, each triad consisting of mutually perpendicular planes; and each triad has its point of intersection situated upon a fixed right line, the directrix of the curve.*

A similar investigation upon plane quartics would lead very likely to a class of twisted cubics whose tangents are three and three mutually perpendicular. The number of conditions to be satisfied is not yet determined.

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* That the two classes are distinct appears from the theorem next the last in § 3.